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EFFECT OF PRESSURE STRESS WORK AND VISCOUS DISSIPATION IN SOME NATURAL CONVECTION FLOWS*

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Abstract—A regular two-parameter perturbation analysis is presented here to study the effects of both viscous dissipation and pressure stress on natural convection flows. Four different vertical flows have been analyzed, those adjacent to an isothermal surface and uniform heat flux surface, a plane plume and flow generated from a horizontal line energy source on a vertical adiabatic surface, or wall-plume. For high gravity levels, stress work effects may be important for gases at very low temperatures, and for high Prandtl number liquids. Significant changes in heat transfer and flow quantities are observed even at moderate values of the perturbation parameters. For the entire range of Prandtl number values considered, the viscous dissipation term is seen to inhibit heat transfer from the surface for heated upward flows. The pressure term enhances heat transfer from the surface for lower Prandtl numbers. However, this effect is seen to reverse at $Pr = 100$, for both the isothermal and uniform flux surfaces. It is observed that viscous dissipation effects on heat transfer are much smaller than those due to the pressure stress, for many practical circumstances.

NOMENCLATURE

b, c, d,	defined in equations (2a)–(2c);
c_p,	specific heat of fluid;
f,	nondimensional stream function;
Gr_x,	local Grashof number in the absence of viscous dissipation and the pressure term $= g\beta x^3(t_o - t_\infty)_0/\nu^2$;
Gr'_x,	actual local Grashof number $= g\beta x^3(t_o - t_\infty)/\nu^2$;
g,	acceleration due to gravity;
h,	local heat transfer coefficient;
k,	thermal conductivity of fluid;
M,	momentum flux in the x-direction;
$\dot{m}$,	mass flow rate per unit width of surface;
N, n,	defined in equations (2a)–(2c);
Nu_x,	local Nusselt number, $= hx/k$;
N',	heat transfer parameter, $= \sqrt{(2)}\, Nu_x/(Gr_x)^{1/4}$;
Q,	total heat convected downstream;
q'',	surface heat flux;
T,	film temperature at which all the fluid properties are calculated;
t,	temperature;
u,	vertical velocity component;
v,	horizontal velocity component;
x,	vertical coordinate;
y,	horizontal coordinate.

Greek symbols

β,	coefficient of thermal expansion;
ε,	perturbation parameter characterizing viscous dissipation, $= g\beta x/C_p$;
λ,	perturbation parameter characterizing pressure stress work, $= (g\beta x/C_p)\, T/\Delta t$
η,	nondimensional horizontal distance;
ν,	kinematic viscosity of fluid;
ρ,	density;
ϕ,	temperature excess ratio, $= (t - t_\infty)/(t_o - t_\infty)$;
ψ,	stream function;
τ,	shear stress.

Subscripts

o	refers to conditions at $x = o$;
∞,	refers to conditions in ambient fluid;
0,	refers to conditions when λ and ε are zero.

INTRODUCTION

IN ALMOST all natural convection studies, the viscous dissipation and pressure stress terms are neglected in the energy equation. This is a valid approximation at an ambient temperature of 300 K at 1 atm pressure and at terrestrial gravity, for most gases and low and moderate Prandtl number liquids. However for high gravity, such as in gas turbine blade cooling applications, where the intensity of the body force may be as large as $10^4\,g$, viscous dissipation and pressure stress effects may affect transport even at small downstream distances from the leading edge. Also, the effects on transport may be quite significant at low temperatures for gases and for high Prandtl number liquids.

Gebhart [1] analyzed the effects of viscous dissipation only, using a regular perturbation analysis. All the previous studies concerning viscous dissipation in natural convection were summarized. The effect of viscous dissipation is obtained in terms of the quantity $g\beta x/c_p$. A fifth-order coupled set of ordinary differential equations is obtained for the first-order

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corrections in velocity and temperature, due to the viscous dissipation effect. Solutions were obtained for various Prandtl numbers ranging from 10^{-2} to 10^4.

Gebhart and Mollendorf [2] consider viscous dissipation effects for an exponentially varying surface temperature. The appropriate parameter for viscous dissipation here was obtained as $g\beta/mc_p$ where m is the e-folding distance for the surface temperature variation. A similar set of equations was obtained for the nondimensional velocity and temperature. The solutions were obtained for various Prandtl numbers. The effect of viscous dissipation on transport was seen to increase with Prandtl number.

Roy generalized the isothermal [3] and uniform surface heat flux [4] results of Gebhart for the case of asymptotically large Prandtl number. In another note [5] Roy also generalized the results of Gebhart and Mollendorf, for asymptotically large Prandtl number. Soundalgekar [6] considered viscous dissipation effects on unsteady natural convection flow past an infinite vertical porous plate with constant suction. Soundalgekar and Pop [7] considered the same problem with non-uniform suction. In follow-up papers [8–12] the viscous dissipation term was retained in the energy equation, for several kinds of transient natural convection. In [13] Soundalgekar considered the effect of mass transfer on free convective flow of an incompressible, dissipative, viscous fluid past an infinite vertical porous plate with constant suction. Effects of viscous dissipation and pressure stress work on mixed convection flow were studied by Soundalgekar and Takhar [14]. The local similarity approach was followed and quasi-ordinary differential equations were obtained for the nondimensional velocity and temperature. These were numerically solved.

Kuiken [15] was the first to consider the effect of pressure stress work in natural convection in gases. The viscous dissipation effect was neglected, however. The set of equations was solved for the case of surface temperature linearly varying with x, for which a similar solution exists. The possible importance of including pressure stress work in plume flow analysis, to describe the outer regions of the plume where the value of $(t_0 - t_\infty)$ is very small, was pointed out.

Ackroyd [16] analyzed stress work and viscous dissipation effects in laminar flat plate natural convection. It was established that pressure work effects are generally more important both for gases and liquids. Property variations within the boundary layer and also outside the boundary layer were considered. Two different kinds of ambient medium property conditions were considered, a constant temperature fluid and isentropic stratification. The surface temperature variations in the two conditions were, $t_0(x) - t_\infty(x) = \text{constant}$ and $t_0 = \text{constant}$, respectively (see the notation). Expansions were made in terms of a perturbation parameter based upon $c_p/g\beta$—the length scale for the viscous dissipation term. However, as we shall see later, the possible x dependence of Δt gives rise in general, to a different length scale for the pressure stress work term. Perturbation solutions were obtained for the nondimensionalized temperature and velocity functions.

Turcotte *et al.* [17] have considered the effect of viscous dissipation on Bénard convection. Hewitt, McKenzie and Weiss [18] examined the energetics of convection in a compressible fluid. They refer to models of convection in the earth's mantle and establish an upper bound to the rate of ohmic heating in the earth's core.

Brown [19], in an integral analysis, examines the relative magnitude of viscous dissipation and pressure stress effects in natural convection over an isothermal vertical flat plate. The von-Karman approximate integral technique has been used with constant fluid properties, both within and out of the boundary layer. Because of a sign error in the substitution for dp/dx, both the pressure stress term and viscous dissipation were found to decrease heat transfer, for heated upward flows.

Gray and Giorgini [20] discuss the validity of the Boussinesq approximation for liquids and gases. Allowance is made for the variation of all properties with temperature and pressure and the explicit ranges where the Boussinesq approximation is valid are found. The fluid properties ρ, c_p, μ, β, and k are assumed linear functions of temperature and pressure. These approximations are substituted into the full set of equations, which are then nondimensionalized. A set of conditions is obtained, for the Boussinesq approximation to be valid. The following two additional conditions are put on the length and temperature difference scales, to justify omitting the effects of viscous dissipation and pressure stress work in the energy equation, respectively

$$\frac{\beta_0 g L T_0}{c_{p_0}(\Delta t)} < 0.1$$

and

$$Pr\frac{\beta_0 g L}{c_{p_0}} < 0.1$$

where the subscript o implies some reference state and L is the length scale. Examples have been given for water and air, at $T_0 = 15°\text{C}$ and $P_0 = 1$ atm. It has been shown that the pressure stress term cannot be neglected in many circumstances. A low value of Δt will make this effect quite important. The viscous dissipation effect is almost always unimportant for water. For air however, it can be inferred that, at low temperatures of 50 K or so, the viscous dissipation effects must be included. The effects arise at large values of x for the terrestrial intensity of gravity. Also, it may be argued from the second condition above, that for liquids with higher Prandtl numbers than water, viscous dissipation may have to be considered. From [20] it can be concluded that for gases at very low reference temperatures and also for high Prandtl number liquids, the viscous dissipation and pressure stress effects can actually become more important than the fluid property variations, both within the boundary layer and in the exterior fluid. The Boussinesq approximations can be invoked then. Clearly, for high values of g, such as in rotating systems, both the viscous dissipation and pressure stress effects will be important even at lower values of x.

The following analysis applies particularly to gases at low temperature levels and to high Prandtl number liquids. Constant fluid properties within the boundary layer and in the ambient medium have been assumed. The Boussinesq approximations have been used. Two different perturbation parameters arise—ε for the viscous dissipation effect and λ for the pressure stress term effect. The equations are determined for a power law variation of the surface temperature. Results have been obtained for four representative kinds of downstream temperature variation. These are—an isothermal surface, a surface dissipating constant heat flux, a plane plume arising from a concentrated horizontal thermal source and an adiabatic surface with a concentrated energy source along the leading edge. The only differences in the formulation for the above four circumstances arise in the boundary conditions and in the coefficients in the relevant differential equations. Expressions for the heat transfer and drag quantities have been found. The results have been obtained for the Prandtl number values of 0.733, 10, and 100.

FORMULATION

This formulation assumes steady, two-dimensional (plane) vertical natural convection flow and incorporates the Boussinesq and boundary layer assumptions. The fluid properties are assumed to be constant, as evaluated at some reference temperature. Viscous dissipation and the hydrostatic pressure terms have been retained in the energy equation. Externally imposed volumetric energy sources are assumed absent. This results in the following governing equations (see for example [21]):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta(t-t_\infty) \tag{1b}$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 t}{\partial y^2}+\frac{\beta T}{\rho c_p}u\frac{\mathrm{d}p_h}{\mathrm{d}x}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{1c}$$

where x is taken to be in the direction of the flow i.e. vertically up from the active leading edge for heated upward flows and vertically down for cooled downward flows. The temperature of quiescent ambient fluid, t_∞, at large values of y, is taken to be constant.

The following generalizations are introduced to obtain the equations in terms of generalized stream and temperature functions f and ϕ

$$\eta(x,y)=yb(x),\ \psi=\nu c(x)f(\eta,x) \tag{2a}$$

$$\phi(\eta,x)=\frac{t-t_\infty}{(t_o-t_\infty)_0},\ (t_o-t_\infty)_0=d(x)=Nx^n \tag{2b}$$

$$c(x)=4xb(x)=4\left[\frac{g\beta x^3(t_o-t_\infty)_0}{4\nu^2}\right]^{1/4} = 4\left[\frac{Gr_x}{4}\right]^{1/4} \tag{2c}$$

where $(t_o-t_\infty)_0$ is the downstream temperature difference (along the x-axis) which would result without the inclusion of the viscous dissipation and hydrostatic pressure effects, that is for both ε and λ zero. Gr_x is related to the actual physical local Grashof number Gr'_x by $Gr'_x = Gr_x\phi(0)$.

Expansions for the stream and temperature functions $f(\eta, x)$ and $\phi(\eta, x)$ are postulated as:

$$f(\eta,x)=f_0(\eta)+\varepsilon(x)f_1(\eta)+\lambda(x)F_1(\eta)+[\varepsilon(x)]^2f_2(\eta)+[\lambda(x)]^2F_2(\eta)+\varepsilon(x)\lambda(x)G_2(\eta)+\dots \tag{3}$$

$$\phi(\eta,x)=\phi_0(\eta)+\varepsilon(x)\phi_1(\eta)+\lambda(x)\Phi_1(\eta)+[\varepsilon(x)]^2\phi_2(\eta)+[\lambda(x)]^2\Phi_2(\eta)+\varepsilon(x)\lambda(x)T_2(\eta)+\dots. \tag{4}$$

To retain both the viscous dissipation and hydrostatic pressure effects to the first order, $\varepsilon(x)$ and $\lambda(x)$ are chosen as

$$\varepsilon(x)=\frac{4g\beta x}{c_p} \tag{5a}$$

$$\lambda(x)=\frac{4g\beta T}{Nc_p}x^{1-n}. \tag{5b}$$

The choice of $\varepsilon(x)$ is the same as made by Gebhart [1]. The quantity $\lambda(x)$ is due to the hydrostatic pressure effect. Note that λ is a constant for $n = 1$. The two quantities are seen to be simply related as

$$\lambda(x)=\varepsilon(x)\frac{T}{d(x)}. \tag{5c}$$

For most practical circumstances $T/d(x)$ is large or at least $O(1)$. Hence $\lambda(x)$ may not be neglected in an investigation of the effect of $\varepsilon(x)$. From 5(a) and (b), x may be eliminated to obtain

$$\varepsilon=[\lambda^{1/(1-n)}]\left[\frac{N}{T}\left(\frac{c_p}{4g\beta}\right)^n\right]^{1/(1-n)} \tag{5d}$$

for heated upward flows. For cooled flows, in the downward direction, the pressure stress causes compression and hence heating of the fluid for $\beta > 0$. This augments heat transfer from the surface. Viscous dissipation also acts to increase heat transfer from surface. Hence in (5d), λ is always positive. However, ε has to be replaced by $-\varepsilon$, for cooled downward flows. The formulation remains the same except that N is replaced by $-N$ and g by $-g$.

Here we consider terms only up to and including the first order. The expansions for f and ϕ are truncated after the first order and λ and ε are treated as separate parameters. This has been done to study the independent effect of each parameter on the velocity and temperature fields. Greater accuracy for specific circumstances may be obtained by retaining higher order terms in (3) and (4). When higher orders terms must be considered, it is convenient to substitute (5d) into (3) and (4), to eliminate $\varepsilon(x)$ in favor of $\lambda(x)$. Depending upon the particular choice of n, (3) and (4) can then be arranged in ascending powers of of λ.

Substituting (3) and (4), into (1b) and (1c), with the generalizations in (2), the equations for f_o, ϕ_o, f_1, ϕ_1, F_1 and Φ_1 are determined for any value of n

$$f_o''' - 2(n+1)f_o'^2 + (n+3)f_of_o'' + \phi_o = 0 \quad (6a)$$

$$\phi_o'' + Pr[(n+3)f_o\phi_o' - 4nf_o'\phi_o] = 0 \quad (6b)$$

$$f_1''' + (n+7)f_o''f_1 + (n+3)f_of_1'' - 4(n+2)f_o'f_1' + \phi_1 = 0 \quad (7a)$$

$$\phi_1'' + Pr[(n+7)f_1\phi_o' + (n+3)f_o\phi_1' - 4(n+1)f_o'\phi_1 - 4nf_1'\phi_o + (f_0'')^2] = 0 \quad (7b)$$

$$F_1''' + (7-3n)f_o''F_1 + (n+3)f_oF_1'' - 8f_o'F_1' + \Phi_1 = 0 \quad (8a)$$

$$\Phi_1'' + Pr[(7-3n)F_1\phi_o' + (n+3)f_o\Phi_1' - 4f_o'\Phi_1 - 4nF_1'\phi_o - f_o'] = 0. \quad (8b)$$

Equations (7a) and (7b) are the same as presented in the Appendix of [2]. A sign error in (7b), which arose in the earlier work, has been corrected.

The relevant boundary conditions, or imposed conditions at $y = 0$ and as $y \to \infty$, are as follows. The primes indicate differentiation with respect to η.

The boundary conditions for the zeroth-order equations are taken to be those that would arise in the absence of viscous dissipation and pressure stress effects. The boundary conditions for the first order terms are then found by imposing reasonable requirements on the velocity and temperature functions f and ϕ, and their derivatives at $\eta = 0$ and at $\eta \to \infty$.

(a) Isothermal surface with horizontal leading edge

$$f_o'(0) = f_1'(0) = F_1'(0) = f_0(0) = f_1(0) = F_1(0) = 0$$
$$f_o'(\infty) = f_1'(\infty) = F_1'(\infty) = 0$$
$$1-\phi_o(0) = \phi_1(0) = \Phi_1(0) = 0$$
$$\phi_o(\infty) = \phi_1(\infty) = \Phi_1(\infty) = 0.$$

(b) Constant flux surface with horizontal leading edge

$$f_o'(0) = f_1'(0) = F_1'(0) = f_o(0) = f_1(0) = F_1(0) = 0$$
$$f_o'(\infty) = f_1'(\infty) = F_1'(\infty) = 0$$
$$1-\phi_o(0) = \phi_1'(0) = \Phi_1'(0) = 0$$
$$\phi_o(\infty) = \phi_1(\infty) = \Phi_1(\infty) = 0.$$

(c) Unbounded plane plume, rising from a horizontal thermal source at $x = 0$

$$f_o(0) = f_1(0) = F_1(0) = f_o''(0) = f_1''(0) = F_1''(0) = 0$$
$$f_o'(\infty) = f_1'(\infty) = F_1'(\infty) = 0$$
$$1-\phi_o(0) = \phi_o'(0) = \phi_1'(0) = \Phi_1'(0) = 0$$
$$\phi_1(\infty) = \Phi_1(\infty) = 0.$$

(d) Adiabatic surface with a concentrated heat source along the horizontal leading edge, a wall plume

$$f_o(0) = f_1(0) = F_1(0) = f_o'(0) = f_1'(0) = F_1'(0) = 0$$
$$f_o'(\infty) = f_1'(\infty) = F_1'(\infty) = 0$$
$$1-\phi_o(0) = \phi_o'(0) = \phi_1'(0) = \Phi_1'(0) = 0$$
$$\phi_1(\infty) = \Phi_1(\infty) = 0.$$

The value of n in (2b), $d = (t_o - t_\infty)_0 = Nx^n$, depends only on the zeroth-order solution. For the isothermal condition $n = 0$, and, therefore, $(t_o - t_\infty)_0$ is given. Thus, the $\phi_1(0)$, $\phi_2(0)$, ... and $\Phi_1(0)$, $\Phi_2(0)$, ... are all zero and the temperature at $y = 0$ does not depend on $\varepsilon(x)$ and $\lambda(x)$. The values of n for the other three flow conditions are determined by calculating the value of $Q_0(x)$, the total heat convected in the flow at any downstream location x, considering only zeroth-order terms.

The energy equation (1c), in the absence of viscous dissipation and pressure stress terms, is integrated at a given x to give

$$Q_0(x) = \int_0^\infty \rho c_p u_0(t-t_\infty)_0\,dy = \int_0^x q_0''\,dx.$$

The subscript 0 emphasizes that viscous dissipation and pressure stress effects are not being considered.

Using generalizations in (2), (3), and (4)

$$Q_0(x) = \int_0^\infty \rho c_p u_0(t-t_\infty)_0\,dy = \rho\nu c_p cd \int_0^\infty \phi_0 f_0'\,d\eta \propto x^{(3+5n)/4}$$

$Q_0(x)$ must increase linearly with x for the imposed uniform surface heat flux condition (b). It must be independent of x for the adiabatic flows, (c) and (d). Therefore,

$$n_a = 0 \quad (9a)$$
$$n_b = 1/5 \quad (9b)$$
$$n_c = n_d = -3/5. \quad (9c)$$

The effects of viscous dissipation and the pressure terms act to generate thermal energy in the flow field. Even in the plume flows, the total convected energy does change downstream of $x = 0$. Including the zeroth- and first-order terms, $Q(x)$ is, in general

$$Q(x) = \rho\nu c_p cd\left[\int_0^\infty \phi_o f_o'\,d\eta + \varepsilon(x) \times \int_0^\infty (f_o'\phi_1 + f_1'\phi_o)\,d\eta + \lambda(x) \times \int_0^\infty (f_o'\Phi_1 + F_1'\phi_o)\,d\eta\right]. \quad (10)$$

The second and third integrals in (10) represent the energy contributions of the viscous dissipation and the pressure terms, respectively, to the total convected energy at location x. The dissipation term in the energy equation (10) is a volumetric source term. The pressure term occurs as a volumetric sink term since $dp_h/dx = -\rho g$, for upward, or heated flows. By integrating the first order energy equations (7b) and (8b) at any x, for conditions b, c, and d, the last two integrals in (10) are evaluated and $Q(x)$ is again written

$$I_{Q_a} = \int_0^{\infty} (f'_o \phi_1 + f'_1 \phi_o)\, d\eta = \frac{1}{(5n+7)} \int_0^{\infty} (f''_o)^2\, d\eta \tag{11}$$

$$I_{Q_b} = \int_0^{\infty} (f'_o \Phi_1 + F'_1 \phi_o)\, d\eta = -\frac{1}{(n+7)} [f_o(\infty)] \tag{12}$$

$$Q(x) = \rho v c_p c d [I_{Q_o} + \varepsilon(x) I_{Q_a} + \lambda(x) I_{Q_b}] \tag{13}$$

where

$$I_{Q_o} = \int_0^{\infty} \phi_o f'_o\, d\eta.$$

The local total downstream mass flow rate, per unit width, is

$$\dot{m} = \int_0^{\infty} \rho u\, dy = v \rho c [f_o(\infty) + \varepsilon(x) f_1(\infty) + \lambda(x) F_1(\infty)]. \tag{14}$$

The local x-direction momentum flux is given by

$$M(x) = \int_0^{\infty} \rho u^2\, dy = \rho v^2 c^2 b \int_0^{\infty} (f')^2\, d\eta$$
$$= \rho v^2 c^2 b [I_{M_o} + \varepsilon(x) I_{M_1} + \lambda(x) I_{M_2}] \tag{15}$$

where

$$I_{M_o} = \int_0^{\infty} (f'_o)^2\, d\eta,\ I_{M_1} = \int_0^{\infty} 2 f'_o f'_1\, d\eta$$

and

$$I_{M_2} = \int_0^{\infty} 2 f'_o F_1\, d\eta.$$

From (5a) and (5b) it is seen that $\varepsilon(x)$ is a function of x for all values of n, as is $\lambda(x)$ for $n \neq 1$, which corresponds to a linearly varying surface temperature. For all the surface conditions considered here, the coordinate expansions (3) and (4) apply. For all four conditions, the exponent of x is equal to or greater than zero in all terms in (5a) and (5b). Hence both expansions are valid for small x. Also, the effect of both the viscous dissipation and pressure terms increases with increasing x.

For surface conditions in (a), (b) and (d) above, the shear stress at the surface, retaining terms up to first order, is given by

$$\tau(x) = \rho v^2 c b^2 [f''_0(0) + \varepsilon(x) f''_1(0) + \lambda(x) F''_1(0)]. \tag{16}$$

Also, the surface heat flux, $q''(x)$, and local Nusselt number, Nu_x, are determined as

$$q'' = -k \left.\frac{\partial t}{\partial y}\right|_{y=0} = [-\phi'(0)]\, k d b$$
$$= [-\phi'(0)] \frac{k(t_0 - t_\infty)_0}{x} \left[\frac{Gr_x}{4}\right]^{1/4} \tag{17}$$

$$Nu_x = \frac{h_x x}{k} = \frac{q''}{(t_0 - t_\infty)_0} \left(\frac{x}{k}\right)$$
$$= -\frac{[\phi'(0)]}{[\phi(0)]^{5/4}} \frac{(Gr'_x)^{1/4}}{\sqrt{2}} \tag{18}$$

where

$$\phi(0) = \phi_o(0) + \varepsilon(x)\, \phi_1(0) + \lambda(x)\, \Phi_1(0) + \ldots$$
$$= \frac{t - t_\infty}{(t_o - t_\infty)_o} \tag{19a}$$

and

$$Gr'_x = Gr_x \phi(0). \tag{19b}$$

Defining N'

$$\frac{Nu_x \sqrt{2}}{(Gr'_x)^{1/4}} = N'$$

(18) is rewritten as

$$N' = \frac{[-\phi'(0)]}{[\phi(0)]^{5/4}}. \tag{20}$$

CALCULATIONS

The zeroth-order equations (6a) and (6b), with suitable boundary conditions and the appropriate value of n for conditions (a), (b) and (c) are written in terms of the formulation of Gebhart [21]. The zeroth order formulation for condition (d) is that of Jaluria and Gebhart [22]. This latter is the flow above a horizontal line source on an adiabatic vertical surface.

Equations (6a) through (8b), with the relevant boundary conditions (a)–(d) were solved numerically for each of the four conditions in (a), (b), (c) and (d), for Pr = 0.733, 10 and 100. Hamming's predictor corrector scheme was used for integration. Initial guesses were corrected using a Taylor series expansion, evaluated at η_{edge}, for the distant boundary conditions. The numerical integration scheme employed automatic local subdivision of the prescribed integration interval, to achieve the desired accuracy. An accuracy criterion of 10^{-10} was used on the distant boundary conditions. The value η_{edge} was increased to as large as 55 to ensure that all results were unvarying up to the fifth digit beyond the decimal point, with further increase in η_{edge}. The Prandtl numbers of 0.733, 10 and 100 represent common gases and many liquids.

RESULTS

The numerical results of the perturbation analysis for the four conditions are collected in Table 1, for the three Prandtl number values. Figure 1 shows f'_0, ϕ_0, f'_1, ϕ_1, F'_1, and Φ_1, for the isothermal surface condition (a), with Pr = 0.733. The correction function due to

Table 1. Calculated flow and transport quantities for various Prandtl numbers and surface conditions

(a) Isothermal surface $n=0$

Pr	$f''_o(0)$	$f''_1(0)$	$F''_1(0)$	$f_o(\infty)$	$f_1(\infty)$	$F_1(\infty)$	$\phi'_o(0)$	$\phi'_1(0)$	$\Phi'_1(0)$
0.733	0.67418	0.00585	−0.03750	0.59434	0.00409	−0.07046	−0.50791	0.07283	−0.12935
10	0.41920	0.00706	−0.07770	0.24923	0.00313	−0.28842	−1.16898	0.13714	−0.21743
100	0.25169	0.00521	−0.14602	0.13664	0.00195	−1.65083	−2.19137	0.46418	0.24668

(b) Uniform flux surface $n=1/5$

Pr	$f''_o(0)$	$f''_1(0)$	$F''_1(0)$	$f_o(\infty)$	$f_1(\infty)$	$F_1(\infty)$	$\phi'_o(0)$	$\phi_1(0)$	$\Phi'(0)$
0.733	0.63751	0.02906	−0.08689	0.54397	0.00577	−0.08371	−0.57945	0.06398	−0.12731
10	0.39503	0.02596	−0.09982	0.33719	0.00391	−0.32231	−1.31642	0.08681	−0.08895
100	0.23668	0.01789	−0.13438	0.12448	0.00243	−1.83758	−2.45844	0.09876	0.11198

(c) Plane plume $n = -3/5$

Pr	$f'_o(0)$	$f'_1(0)$	$F'_1(0)$	$f_o(\infty)$	$f_1(\infty)$	$F_1(\infty)$	I_{M_o}	I_{M1}	I_{M2}	$\phi_1(0)$	$\Phi_1(0)$	I_{Q_o}	$\int_0^\infty (f''_o)^2\, d\eta$
0.733	0.65727	0.00948	−0.02752	0.91638	0.00985	−0.03579	0.40500	0.01095	−0.03066	0.04450	−0.18714	0.59282	0.16572
10	0.41395	0.01128	−0.04048	0.50156	0.00884	−0.06829	0.12735	0.00682	−0.02900	0.03984	−0.20851	0.15387	0.07267
100	0.25066	0.00801	−0.06745	0.36253	0.00704	−0.17023	0.05085	0.00333	−0.03939	0.01564	−0.18529	0.03972	0.02130

(d) Plane wall plume $n=-3/5$

Pr	$f''_o(0)$	$f''_1(0)$	$F''_o(0)$	$f_o(\infty)$	$f_1(\infty)$	$F_1(\infty)$	I_{M_o}	I'_{M1}	I_{M2}	$\phi_1(0)$	$\Phi_1(0)$	I_{Q_o}	$\int_0^\infty (f''_o)^2\, d\eta$
0.733	0.92423	0.10580	−0.10599	0.87368	0.01216	−0.04099	0.26891	0.01527	−0.02779	0.22212	−0.18840	0.53078	0.33888
10	0.61060	0.10321	−0.11460	0.38357	0.00821	−0.14407	0.04899	0.00418	−0.02701	0.33626	−0.19840	0.10676	0.09149
100	0.37679	0.07367	−0.14022	0.21241	0.00518	−0.79706	0.00967	0.00095	−0.04751	0.40098	−0.15041	0.02101	0.01989

viscous dissipation is seen to be smaller in magnitude than that due to the pressure stress term. Also, it does not extend as far out into the boundary region. The governing equations, for f_1 and ϕ_1, and the boundary conditions for this condition, are the same as in Gebhart's [1] analysis for viscous dissipation. However, for these latter solutions, the approximation $= 0$ was made to simplify the calculations. This eliminates equation (7a) completely. Further, in (7b) two terms drop out, making it simpler to solve. The resulting value for $\phi_1'(0)$ is 5% different from the more accurate results obtained here, for $Pr = 100$. None of the other Prandtl number values used here are common to both analyses.

It is seen in Table 1 that, for both $Pr = 0.733$ and 10, $\Phi_1'(0)$ is negative. For heated upward flows this means that the pressure stress effect increases the heat transfer from the surface. However, this effect is reversed at $Pr = 100$. Looked at in another way, for $Pr = 0.733$ and 10, Φ_1 is negative over the boundary region and for $Pr = 100$ it is positive in the region $0 \leq \eta \lesssim 0.5$ and negative for $\eta \gtrsim 0.5$. Thus, the pressure term actually does inhibit heat transfer from the surface in a heated upward flow. This is explained as the effect of the simultaneous interplay of two factors; on the one hand, there is a downstream tendency of fluid cooling due to its expansion. On the other hand, there is a decrease in the heat convected, due to the reduction in the velocity level resulting from the reduced buoyancy force. The latter effect becomes the dominant one near the surface, at higher Prandtl numbers. The pressure term then decreases surface heat transfer and actually causes relative local heating of the fluid.

Figure 2 presents f_0', ϕ_0, f_1', ϕ_1, F_1' and Φ_1, for the uniform imposed flux condition, with $Pr = 0.733$. Here ϕ_1 is greater in magnitude and extends out further than for the isothermal condition. However, f_1' and F_1' are still small. A weak reversal in f_1' is observed for $\eta \gtrsim 2$.

The effect of λ and ε on the heat transfer parameter N' in (20), is shown in Fig. 3, for both the isothermal and uniform flux conditions, for $|N|/T = 0.1$. As explained later, physical considerations limit the permissible values of λ and ε within a range. The region between A and A' in Fig. 3 denotes the region of applicability of the present analysis. The viscous dissipation effect always generates frictional heat and acts as a source term, both for heated upflow or cooled downflow. The pressure stress term however becomes an energy source term for cooled downward flows, since $dp_h/dx > 0$. The fluid is actually being compressed and thereby warmed. In (5b) the ratio g/N remains positive both for heated upflow and cooled downflow. Therefore, in Fig. 3, λ is always positive. Also, ε is positive for heated upflow and negative for cooled downflow, because of the change in the sign of g. The value of λ for given ε is obtained from (5d). For heated upflow $(t_o - t_\infty)_0 = d > 0$, $N > 0$. For cooled downflow $N < 0$. As mentioned earlier, x always

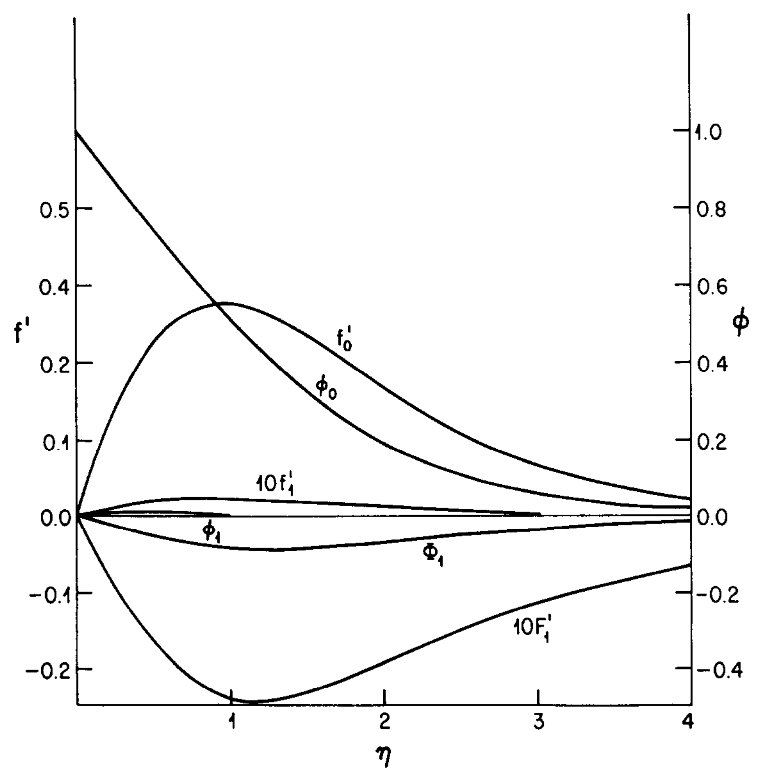

FIG. 1. Temperature and velocity functions for the isothermal condition (a) with $Pr = 0.733$, ϕ_o, ϕ_1, Φ_1, f_0', f_1', and F_1'.

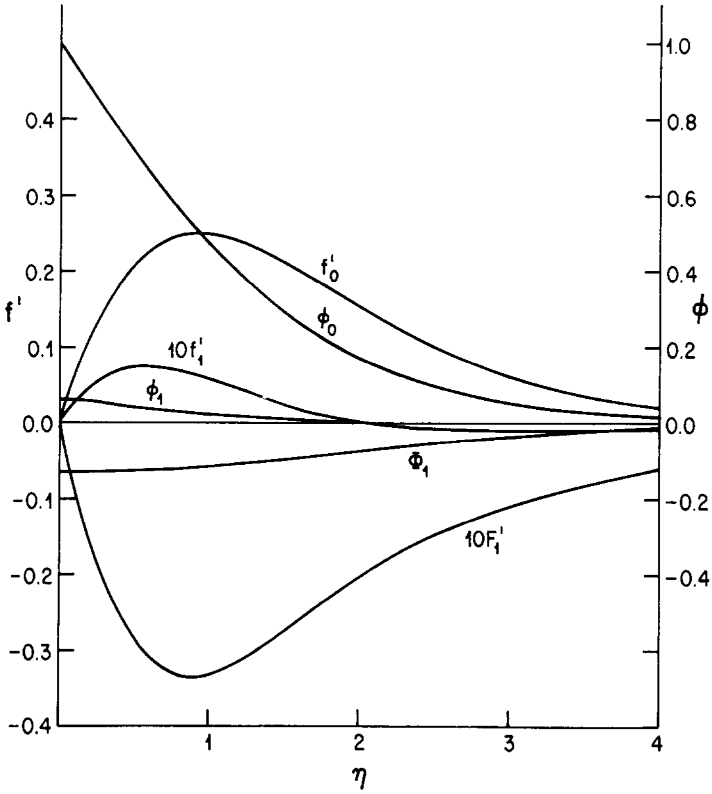

FIG. 2. Temperature and velocity functions for the uniform flux condition (b) with $Pr = 0.733$, ϕ_o, ϕ_1, Φ_1, f'_0, f'_1, and F'_1.

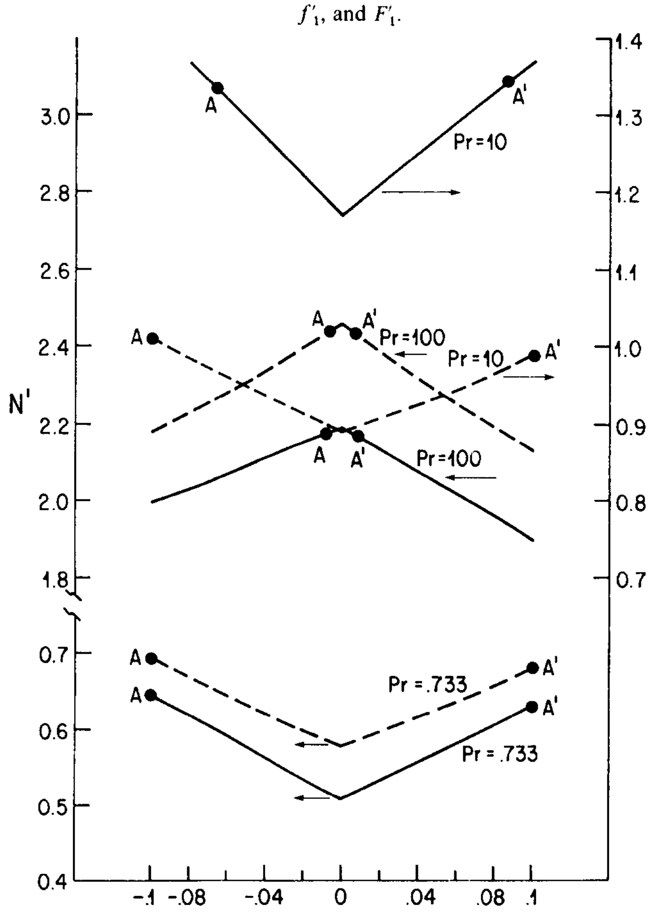

FIG. 3. The effect of ε and λ for various Prandtl numbers, on heat transfer for isothermal (——) and uniform flux (---) surface. $|N|/T = 0.1$ has been assumed for all the curves. The region between A and A' gives the range of validity of results without encountering $f(x) \leq 0$.

remains positive. Figure 3 indicates that, for Pr = 0.733 and 10, increasing the value of ε from 0 in either direction, increases N' for both the isothermal and uniform flux condition. For example, for $\varepsilon = 0.08$, $\lambda = 0.8$ and $Pr = 10$, N' is seen to increase by 13.7% for isothermal surface and 8.6% for constant flux surface, over its value at $\varepsilon = 0$. For $Pr = 100$ the trend in the variation of N' changes, due to the change in sign of $\Phi_1'(0)$ and $\Phi_1(0)$ for the two surface conditions, respectively.

Results in Table 1 indicate that the value of $-F_1(\infty)$ rises very sharply as Pr is increased. The boundary layer thickness $\delta(x)$ must increase with x, from $\delta = 0$ at the leading edge. For this to be true, fluid must be entrained from the ambient and $v(x, \infty) < 0$, which in the transformed variables implies that $f(\infty) > 0$. The conditions for $f(\infty) > 0$, for $Pr = 100$ and $|N|/T = 0.1$ are calculated to be $\lambda \leq 0.083$ and $\lambda \leq 0.068$, for the isothermal and uniform flux conditions for heated upflow. Thus, pressure stress effects become quite significant at high Prandtl number. Similar limits could be obtained for each Prandtl number and value of the parameter $|N|/T$. It is also noted that, since ε is typically much smaller than λ, the pressure stress term influences heat transfer much more than does viscous dissipation.

Figure 4 shows f_0', ϕ_0, f_1', ϕ_1, F_1', and Φ_1 for the plane plume with $Pr = 0.733$. Again, the correction due to the pressure term is greater than that due to dissipation. As for the uniform flux condition, there is a weak reversal in f_1' for $\eta \geq 2.25$. Table 1 indicates that $\Phi_1(0)$ is less for $Pr = 100$ than for $Pr = 10$. This is again attributed to the large decrease in velocity level, the decrease in the convected energy in the plume, for a higher Prandtl number. Nonzero values of λ and ε produced significant changes in the velocity and temperature at the plume centerline. This is shown in Fig. 5. The effect of increasing Prandtl number on $\phi_1(0)$, seen in Table 1, is a decrease. This trend is not observed for the other three conditions.

The solutions for the wall plume are shown in Fig. 6, for $Pr = 0.733$. The values of ϕ_1 and f_1' are seen to be larger for this than for the previous three conditions. The importance of the presence of a surface, in making viscous dissipation appreciable, is seen by comparing the wall plume values of $\phi_1(0)$, with that for an unbounded plane plume (see Table 1). It is clear that the presence of the surface shear enhances the frictional heating effect considerably. The viscous dissipation corrections in this condition, are of the same magnitude as the corrections due to the pressure term. As for the uniform flux surface and the plane plume, a reversal in f_1' is observed for $\eta \gtrsim 1.7$. A very small reversal in F_1' is observed for $2.4 \lesssim \eta \lesssim 3$.

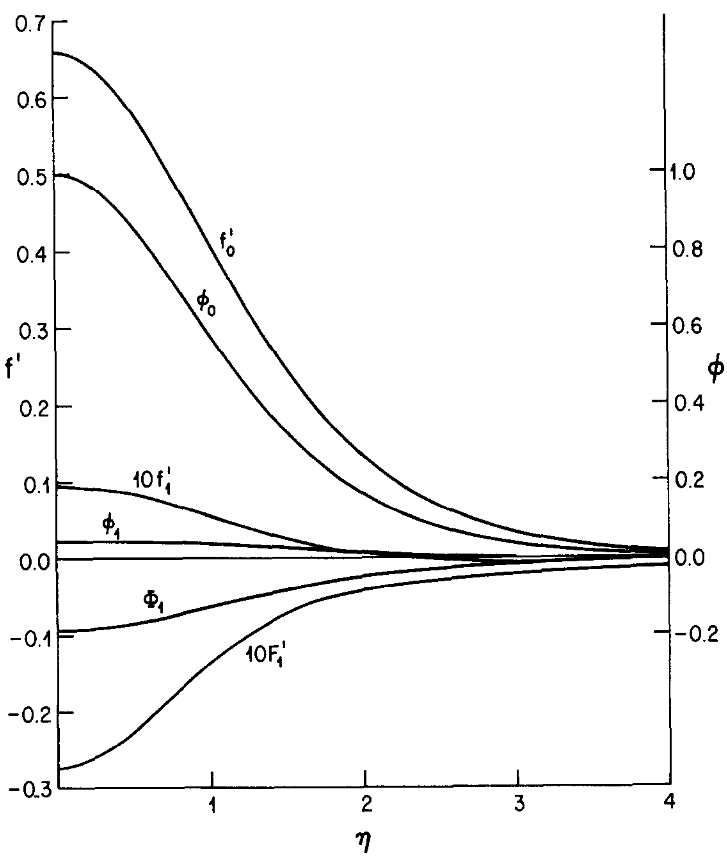

FIG. 4. Temperature and velocity functions for the plane plume (c) with $Pr = 0.733$, ϕ_0, ϕ_1, Φ_1, f_0', f_1', and F_1'.

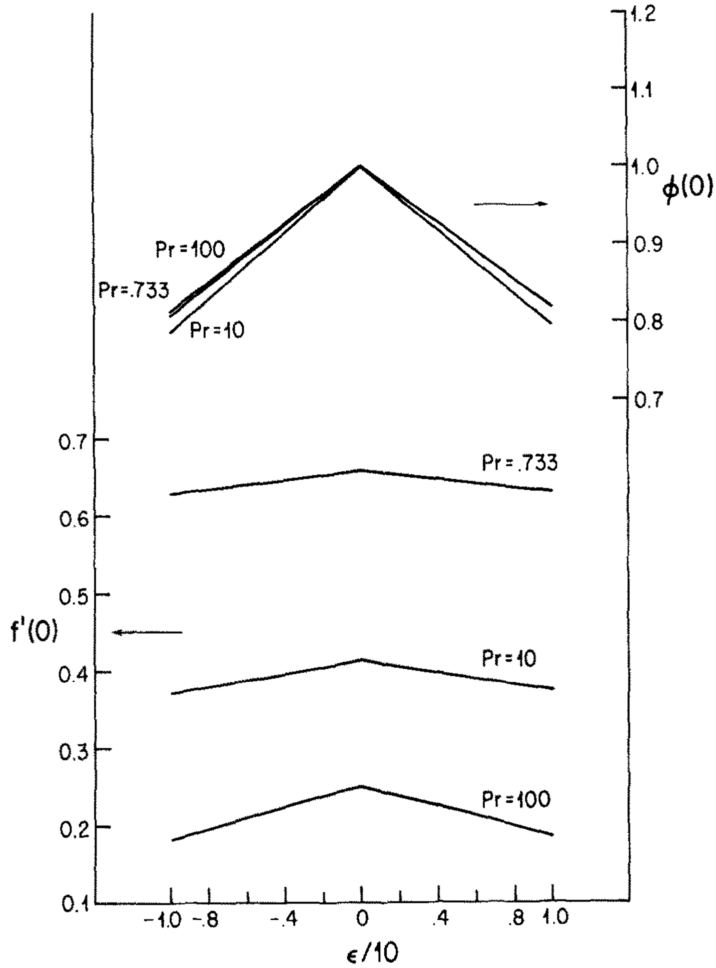

FIG. 5. The effect of λ, ε and Prandtl number on $\phi(0)$ (upper curves) and $f'(0)$ (lower curves) for the plane plume. $|N|/T = 0.1$ has been used. The effect of λ arises implicitly in the expression for $\phi(0)$.

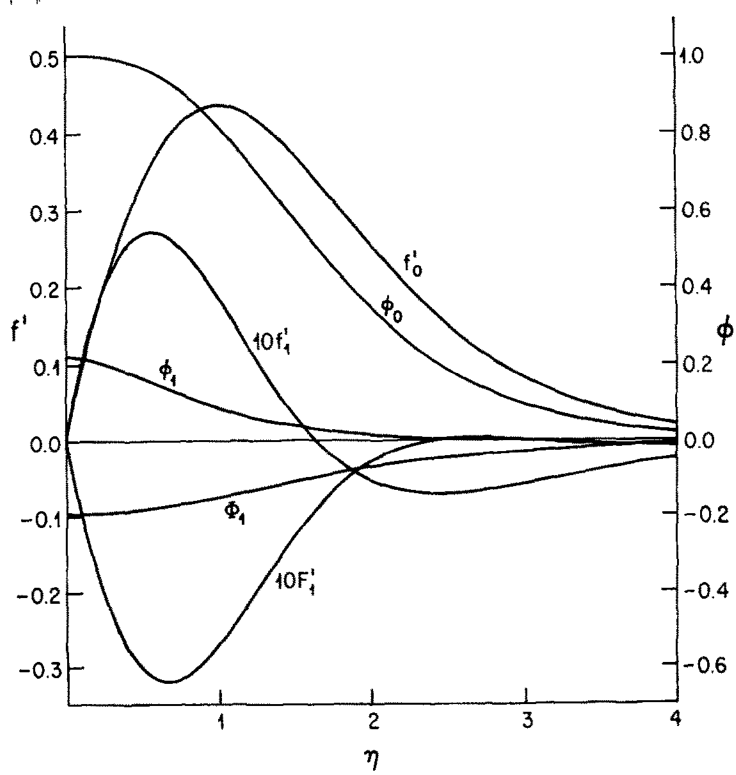

FIG. 6. Temperature and velocity functions for the wall, plume (d) with $Pr = 0.733$, ϕ_o, ϕ_1, Φ_1, f'_1, and F'_1.

CONCLUSION

This analysis considers both the viscous dissipation and pressure stress work for various types of surface temperature variations, in a unified manner. For the case of high gravity, stress work effects may be important for gases at low temperatures and for high Prandtl number liquids. This analysis provides the correct estimate of heat transfer and fluid flow quantities for such circumstances. Four representative surface conditions have been considered. The effects have been studied for three different Prandtl numbers for each surface condition. Viscous dissipation and pressure stress effects have been retained as first order effects. The resulting three sets of coupled fifth-order ordinary differential equations have been solved numerically. It is observed that the pressure stress term has a much greater effect than viscous dissipation, on heat transfer, for all the four surface conditions analyzed. Significant effects on flow and heat transfer were found even for moderate values of ε and λ. These effects are seen to be greatest for the two plume flows.

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EFFET DE LA PRESSION ET DE LA DISSIPATION VISQUEUSE DANS QUELQUES ECOULEMENTS DE CONVECTION NATURELLE

Résumé—On présente une analyse de perturbation à deux paramètres pour étudier en même temps les effets de la dissipation visqueuse et de la pression sur la convection naturelle. On considère quatre écoulements verticaux, ceux adjacents à une surface isotherme et à une surface à flux uniforme, un panache plan, et un écoulement issu d'une source d'énergie linéaire horizontale sur une surface adiabatique verticale, ou panache pariétal. Pour des niveaux de pesanteur élevés, l'effet du travail des tensions peut être important pour les gaz à des températures très basses et pour les liquides à grand nombre de Prandtl. On observe des changements sensibles dans le transfert de chaleur et les débits de fluide, même à des valeurs modérées du paramètre de perturbation. Pour le domaine entier du nombre de Prandtl considéré, le terme de dissipation visqueuse freine le transfert thermique à la surface, pour les écoulements ascendants. Le terme de pression accroit le transfert thermique pour les nombres de Prandtl faibles. Néanmoins cet effet se renverse à $Pr = 100$, à la fois pour les surfaces isothermes ou à flux constant. On constate que l'effet de la dissipation visqueuse sur le transfert thermique est plus faible que celui de la pression dans beaucoup de circonstances pratiques.

DER EINFLUSS DER DRUCKARBEIT UND DER VISKOSEN DISSIPATION AUF EINIGE NATÜRLICHE KONVEKTIONSSTRÖMUNGEN

Zusammenfassung—Es wird hier eine reguläre zweiparametrige Störungsanalyse dargestellt, um den Einfluß sowohl der viskosen Dissipation als auch der Druckarbeit auf natürliche Konvektionsströmungen zu untersuchen. Vier verschiedene senkrechte Strömungen werden untersucht, anliegende Strömungen an einer isothermen und an einer Fläche mit konstanter Wärmestromdichte, eine ebene Auftriebsströmung und eine Strömung, die von einer waagerechten linienförmigen Energiequelle an einer senkrechten adiabaten Wand ausgeht, eine sogenannte Wandauftriebsströmung. Bei hohen Werten der Schwerkraft ist die Druckarbeit für Gase bei sehr niedrigen Temperaturen und für Flüssigkeiten mit großen Prandtl–Zahlen von Bedeutung. Signifikante Änderungen der Wärmeübertragung und der Strömungsgrößen werden sogar bei mäßigen Werten des Störungsparameters beobachtet. Für den gesamten Bereich der untersuchten Prandtl–Zahlen erkennt man, daß der viskose Dissipationsterm die Wärmeübertragung von der Wand bei beheizten Aufwärtsströmungen behindert. Der Druckterm begünstigt die Wärmeübertragung von der Wand bei kleinen Prandtl–Zahlen. Es zeigt sich jedoch, daß sich dieser Effekt bei $Pr = 100$ umkehrt, und zwar sowohl bei isothermen Flächen als auch bei Flächen mit konstanter Wärmestromdichte. Es wird beobachtet, daß viskose Dissipationseinflüsse auf die Wärmeübertragung in vielen praktischen Fällen sehr viel geringer als die Einflüsse der Druckarbeit sind.

ВЛИЯНИЕ РАБОТЫ УПРУГИХ НАПРЯЖЕНИЙ И ВЯЗКОЙ ДИССИПАЦИИ НА НЕКОТОРЫЕ ТИПЫ ЕСТЕСТВЕННОКОНВЕКТИВНЫХ ТЕЧЕНИЙ

Аннотация — С помощью двухпараметрического метода возмущений исследуется влияние вязкой диссипации и нормальных напряжений на естественноконвективное течение. Анализируется четыре вида восходящих потоков: у изотермической поверхности, у поверхности с постоянным тепловым потоком, плоская струя и поток от горизонтального линейного источника тепла, расположенного на вертикальной адиабатической поверхности, или восходящая струя вблизи стенки. При больших значениях силы тяжести работа напряжений может оказывать существенное влияние на течение газов при очень низких температурах и жидкостей с большим числом Прандтля. Даже при небольших значениях параметра возмущения наблюдаются большие изменения в плотности теплового потока и картине течения. Показано, что в случае нагреваемых восходящих потоков во всем диапазоне рассматриваемых значений числа Прандтля слагаемое, описывающее вязкую диссипацию, учитывает величину теплового потока от поверхности. При более низких значениях числа Прандтля давление оказывает интенсифицирующее воздействие на теплоперенос от поверхности. Однако при $Pr = 100$ наблюдается противоположный эффект как для изотермических, так и равномерно нагреваемых поверхностей. Найдено, что в большинстве практически важных случаев влияние нормальных напряжений на теплоперенос намного превосходит влияние вязкой диссипации.